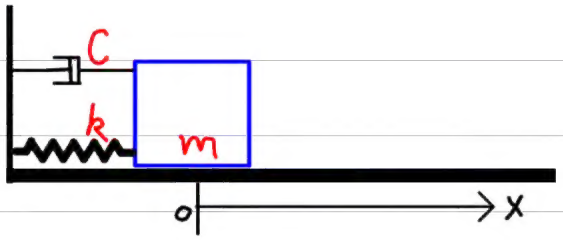
Meriam 8/2: Damped Vibrations – Sample Problems

Solution Technique



1/ Set origin at neutral point (damper has no effect)
Select positive direction

2/ Displace mass a small +ve distance x, and give it a small +ve velocity $\dot{x}$ Release the mass.

3/ $\sum F_x = -kx - c\dot{x} = m\ddot{x}$ (For our simple example)

4/ $\rightarrow$ $\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$$

$\left\{ \begin{array}{l} \omega_n^2 = k/m \rightarrow \omega_n \\ \frac{c}{m} = 2\zeta\omega_n \rightarrow \zeta \\ \zeta = \frac{c}{c_c} \rightarrow c_c \end{array} \right.$

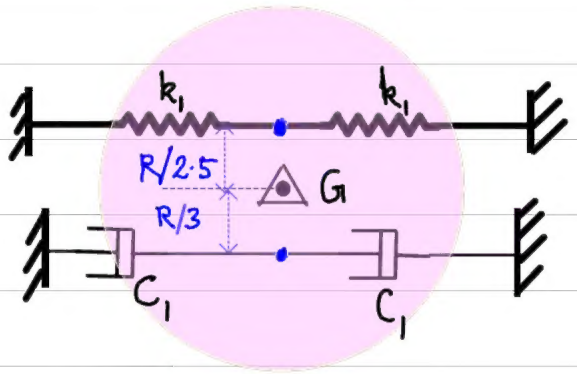
ζ = Damping factor (damping ratio)

- Define $c_{critical}$ (or c_c) to be value of c to make $\zeta = 1$
- $\rightarrow$ $\zeta = \frac{c}{c_c}$
- For underdamped case where $\zeta < 1$,

$$x(t) = A\sin(\omega_d t + \phi)\exp(-\zeta\omega_n t), \quad \omega_d = \omega_n\sqrt{1-\zeta^2}$$

Eg.

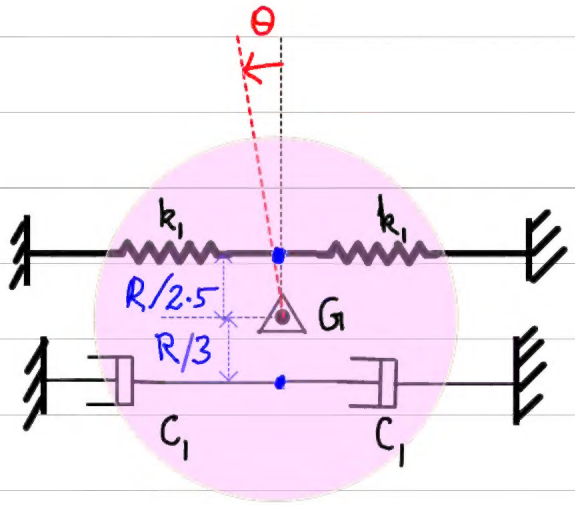
Given: k_1, C_1, radius R, mass M_{disk},
Radius of gyration k_G
Neutral position as shown
Pinned at center G



Find: ω_n, ω_d, ζ, $\theta(t)$

1/ This is a rotational vibration, with θ as independent variable

2/ Give the disk a small +ve deflection θ, & small +ve angular velocity $\dot{\theta}$. (+↺)
Then release it.



3/ $\sum M_G = I_G \alpha$

$$-\frac{R}{2.5}\left[(k_1)\left(\frac{R}{2.5}\theta\right)\right]-\frac{R}{2.5}\left[(k_1)\left(\frac{R}{2.5}\theta\right)\right]-\frac{R}{3}\left[(C_1)\left(\frac{R}{3}\dot{\theta}\right)\right]-\frac{R}{3}\left[(C_1)\left(\frac{R}{3}\dot{\theta}\right)\right]=M_{disk}\,k_G^2\,\ddot{\theta}$$

$$\rightarrow \quad M_{disk}\,k_G^2\,\ddot{\theta} + 2C_1\left(\frac{R}{3}\right)^2\dot{\theta} + 2k_1\left(\frac{R}{2.5}\right)^2\theta = 0$$

$$\ddot{\theta} + \frac{2C_1\left(\frac{R}{3}\right)^2}{M_{disk}\,k_G^2}\dot{\theta} + \frac{2k_1\left(\frac{R}{2.5}\right)^2}{M_{disk}\,k_G^2}\theta = 0$$

$$\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = 0$$

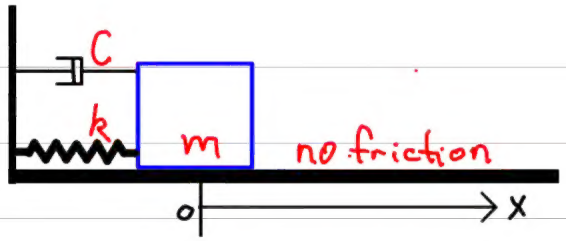
$$\rightarrow \quad \omega_n = \sqrt{\frac{2k_1\left(\frac{R}{2.5}\right)^2}{M_{disk}\,k_G^2}} \qquad \zeta = \left[\frac{2C_1\left(\frac{R}{3}\right)^2}{M_{disk}\,k_G^2}\right] \Big/ 2\omega_n$$

$$\omega_d = \omega_n \sqrt{1-\zeta^2}$$

$$x(t) = A \sin(\omega_d t + \phi) \exp(-\zeta \omega_n t)$$

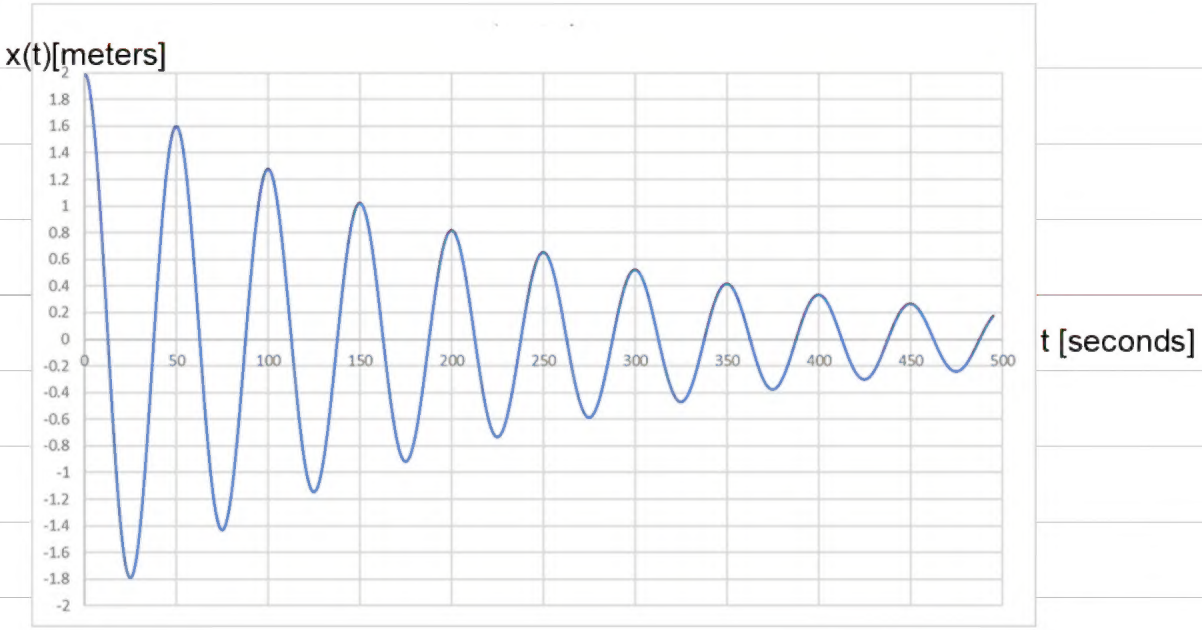
Eg.

Mass m is displaced 2 meters ↑ and released. The data shown in the chart were recorded.



Find: $\omega_n, \omega_d, \zeta$

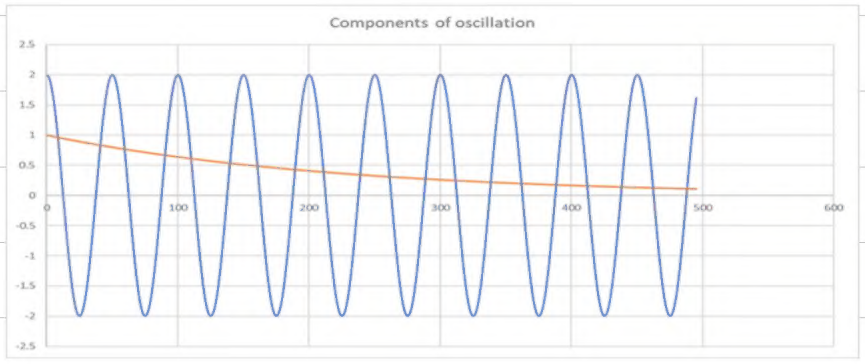
Estimate A, ϕ in $x(t) = A \sin(\omega_d t + \phi) \exp(-\zeta \omega_n t)$



From graph, $T_D \approx 50\,s \rightarrow \omega_d = \frac{2\pi}{T_D} = 0.1257\ s^{-1}$

From graph, $A \approx 2$ meters

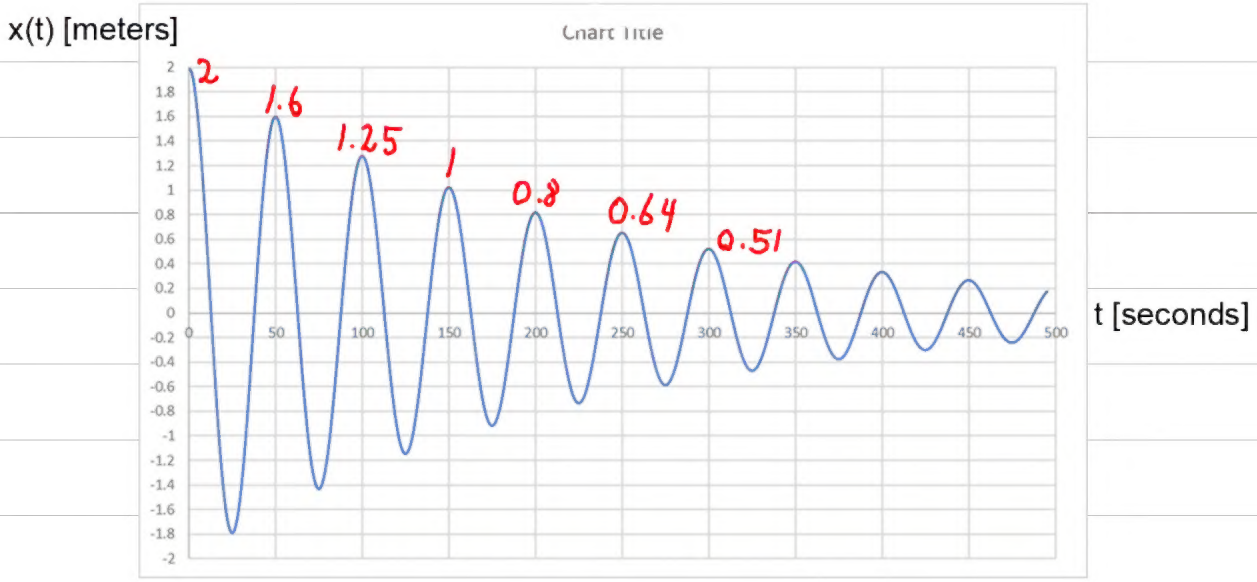
From graph, $\phi \approx 90^\circ$



ζ ?

Note graph is the product of 2 functions:

$A \sin(\omega_d t + \phi)$ (A: 2 m, ω_d: 0.1257 s^{-1}, ϕ: $\pi/2$) & $\exp(-\zeta \omega_n t)$



In each period $\tilde{\tau}_d$, the decaying exponential causes the oscillation amplitude to decrease by a factor of ~0.8.

$$\rightarrow \quad \ln\left\{ \exp(-\zeta \omega_n \tilde{\tau}_d) = 0.8 \right\}$$

$$-\zeta \omega_n \tilde{\tau}_d = \ln(0.8) \quad \rightarrow \quad \boxed{-\frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \ln(0.8)} \quad \star$$

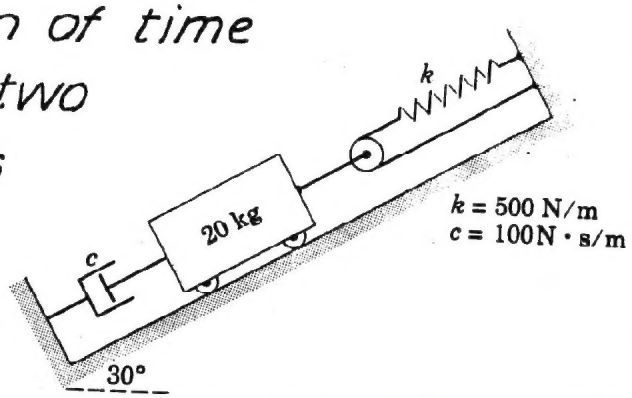
($\omega_n = \omega_d/\sqrt{1-\zeta^2}$; $\tilde{\tau}_d = 2\pi/\omega_d$)

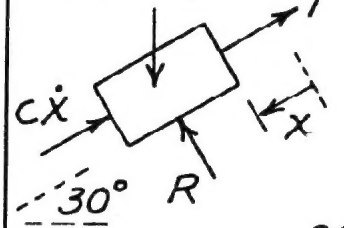
$$\rightarrow \quad \zeta/\sqrt{1-\zeta^2} = 0.03551$$

$$\rightarrow \quad \boxed{\zeta = 0.0355} \quad ; \quad \boxed{\omega_n = \omega_d/\sqrt{1-\zeta^2} = 1.00063\,\omega_d = 0.12574\ s^{-1}}$$

ART. 8/2 FREE VIBRATION OF PARTICLES (DAMPED) NO. 2

The 20-kg body is displaced a distance of 200 mm down the incline from its equilibrium position and released from rest. Express the displacement x from the equilibrium position as a function of time and find the ratio of two successive amplitudes of the damped oscillation.





Select origin where $\Sigma F_x = 0$ (equilibrium) for equiv. spr. const. $K = 4k = 2000$ N/m)

$\Sigma F_x = m\ddot{x}$ gives

$$-2000x - 100\dot{x} = 20\ddot{x}$$

$$20\ddot{x} + 100\dot{x} + 2000x = 0, \quad \ddot{x} + 5\dot{x} + 100x = 0$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$$

$\omega_n = \sqrt{100} = 10$;

$2\zeta\omega_n = 5 \longrightarrow \zeta = 0.25$

$\omega_d = \omega_n\sqrt{1-\zeta^2} = 10\sqrt{1-(0.25)^2} = 9.68$ rad/s $\quad \tau_D = \frac{2\pi}{\omega_D} = 0.6488$ s

$x = Ce^{-2.5t}\sin(9.68t + \psi)$; $x = 0.2$ m & $\dot{x} = 0$ for $t = 0$ give

$0.2 = C\sin\psi$ & $0 = -2.5C\sin\psi + 9.68C\cos\psi$. Solve & get

$C = 0.207$ m, $\psi = 1.318$ rad so $\boxed{x = 0.207e^{-2.5t}\sin(9.68t + 1.318)}$

$\delta = \ln\left(\frac{x_1}{x_2}\right) = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \frac{2\pi(0.25)}{\sqrt{1-(0.25)^2}} = 1.622$, $\boxed{x_1/x_2 = 5.06}$ $\frac{x_2}{x_1} = 0.198$ JLM

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see eqn ★ on page 36(d)

$$\text{Ratio} = \exp(-\zeta\omega_n\tau_D) = \exp\left(-[0.25][10][0.6488]\right) = 0.198$$

Summary - Damped Vibrations

- Governing equation: $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$

- For underdamped case, $\zeta = \frac{c}{c_c} < 1$,

$$x(t) = A\sin(\omega_d t + \phi)\exp(-\zeta\omega_n t), \quad \omega_d = \omega_n\sqrt{1-\zeta^2}$$

- Rotational Vibrations are based on the equation:

$$\Sigma M_o = I_o\ddot{\theta} \quad , \quad \text{or} \quad \Sigma M_G = I_G\ddot{\theta}$$

Next Lecture

- Read Meriam 8/3: Forced Vibrations